Solving Applied Optimization Problems with Differential Equations

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***Abstract*—** There are many different rules for mathematics field. Among them, solving the problems with differential equations which there are first derivative test and second derivative test. There were use to solve the problem of practical life such as profit and loss of the business system, maximum and minimum of the dimensions and increasing and decreasing of the function domain. In this paper, we use derivatives to find the extreme values of functions, to determine and analyze the shapes of graphs and to find numerically where a function equals zero. Rolle's Theorem and its examples are expressed. The first derivative test to determine where the differential equation increases and decreases is given. Similarly, the second derivative test to determine. Variety of optimization problems are solved by using derivatives. The dimension of various shape with fixed perimeter having maximum area are computed. The dimension for the least expensive cylindrical can of a given volume are computed.

***Keywords*—** Differential Equations, Derivative, Absolute Maximum values, Absolute Minimum values, Local Maximum values, Local Minimum values.

I. INTRODUCTION

This paper shows how to locate and identify extreme (maximum or minimum) values of a function from its derivative. We can solve a variety of problems in which we find the optimal way to do something in a given situation. Finding maximum and minimum value is one of the most important applications of the derivative. First derivative test and second derivative test for local extrema values are expressed.

Roller's Theorem, as suggested by its graph, if a differentiable function crosses a horizontal line at two different points, there is at least one point between them where the tangent to the graph is horizontal and the derivative is zero. How to the first derivative test the critical points of a function to identify whether local extreme values are presented. How to the second derivative test the critical points of a function to identify whether local maximum or local minimum values are computed. Differential equations are basic life appear mathematically in form of a differential equation. We will denote a differential equation as y'=f'(x).

II. SOME BASIC DIFFERENTIAL EQUATIONS

An ordinary differential equation is an equation that contains one or more several derivatives of an unknown function.

For example, $y' = f'(x) = \sin x$.

$$\frac{dy}{dx} = \frac{f(x,y)}{y} \text{ that is } y' = f'(x) = (x+5)e^x.$$

$$y'' = f''(x) = sec^2 x.$$

III. EXTREME VALUES OF A FUNCTION

A. Definition

Let f be a function with domain D. Then f has an **absolute maximum** value on domain D at a point c if

$f(x) \leq f(c)$ for all x in D

and an **absolute minimum** value on domain D at c if

$f(x) \geq f(c)$ for all x in D.

Maximum and minimum values are called **extreme values** of the function f. Absolute maxima or minima are also referred to as **global maxima or minima.**

B. Definition

A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all x in D lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all x in D lying in some open interval containing c.

Local extrema are called **relative extrema**.

C. Definition

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

D. Example

The extreme value of $y = f(x) = cos\ x$ on the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ can be found.

$$y = f(x) = cos\ x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

Differentiate with respect to x in both sides,

$$f'(x) = -\sin x.$$

To find the critical point, $-sin\ x = 0$.

$$sin\ x = 0, \quad x = 0, \pi.$$

Only $x = 0$ lies in the interior of the function domain.

The critical value, $f(0) = cos\, 0 = 1$.

The endpoint value, $f(-\frac{\pi}{2}) = cos\,(-\frac{\pi}{2}) = 0$.

$$f(\frac{\pi}{2}) = cos\,(\frac{\pi}{2}) = 0.$$

Maximum values is 1 at x = 0.

Minimum values is 0 at x $=\pm\frac{\pi}{2}$.

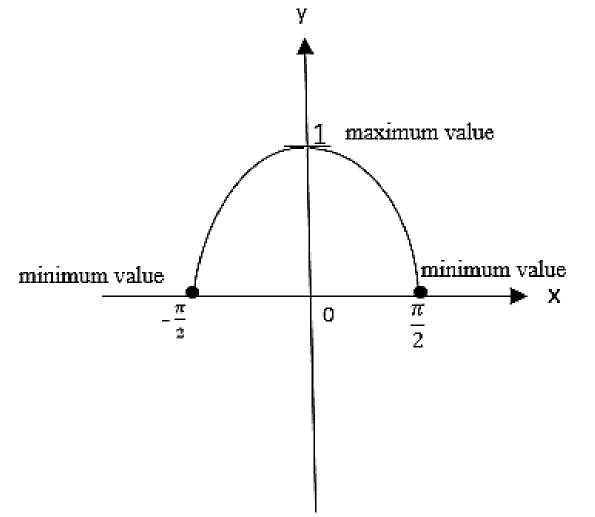

Figure1. Absolute extrema for the cosine function on ($-\frac{\pi}{2}, \frac{\pi}{2}$).

E. Rolle's Theorem

Suppose that y=f(x) is continuous at every point of the closed interval [a, b] and differentiable at every point of its interior (a, b). If f(a)=f(b), then there is at least one number c in (a ,b) at which f '(c)=0.

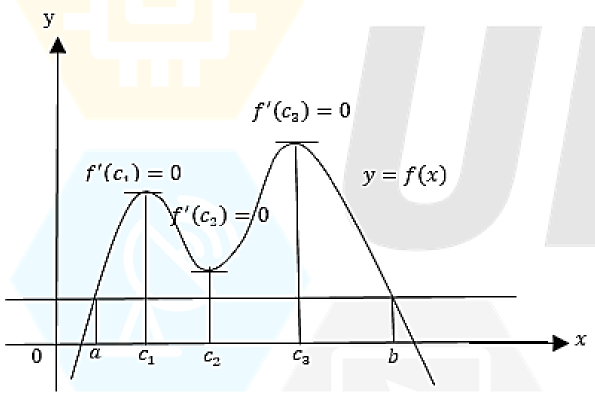

Figure 2. Roller's Theorem say that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line.

F. Example

The equation $x^3 + 3x + 1 = 0$ on open interval (-1,0) has exactly one real solution.

$f(x) = x^3 + 3x + 1$

$f(-1) = -3, \quad f(0) = 1.$

$f\,'(x) = 3x^2 + 3$ is never zero because it is always positive.

Now, if there were even two points x= a and x=b where f(x) was zero, Rolle's Theorem would guarantee the existence of a point in x=c in between then where $f\,'$ was zero. Therefore, f has no more than one zero.

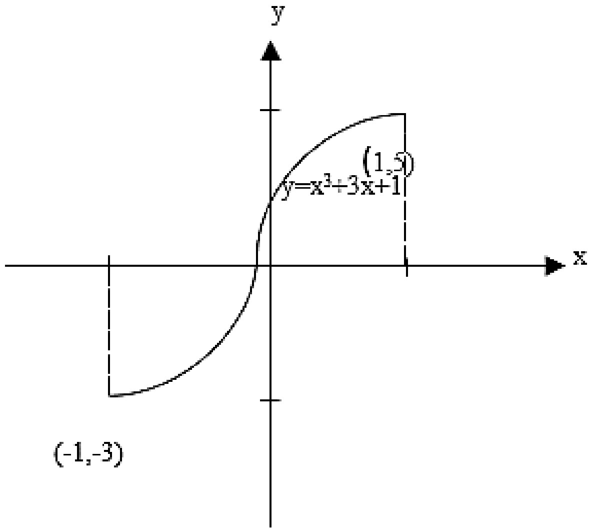

Figure 3. The only real zero of the polynomial y=x3+3x+1 is the one shown here where the curve crosses the x-axis between -1 and 0.

IV. FIRST DERIVATIVE TEST FOR LOCAL EXTREME

Suppose that c is a critical point of a continuous function $\boldsymbol{f}$, and that $\boldsymbol{f}$ is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right.

If $f\,'$ changes from negative to positive at c, then f has a local minimum at c.

If $f\,'$ changes from positive to negative at c, then f has a local maximum at c.

If $f\,'$ does not change sign at c, then f has no local extremum at c.

A. Theorem

The first derivative theorem for local extreme values-If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

Proof: Suppose that f has a local maximum value at $x = c$.

$f(x) \le f(c)$ for all values of x near enough to c.

$f(x) - f(c) \le 0.$

Since c is an interior point of function domain,

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}.$$

$x \to c^+, x - c > 0$ *and* $\quad f(x) - f(c) \le 0.$

$$f'(c) = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \le 0. \qquad (1)$$

$x \to c^-, x - c < 0$ *and* $\quad f(x) - f(c) \le 0.$

$$f'(c) = \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \ge 0. \qquad (2)$$

By the equation (1) and (2), $f'(c) = 0$.

Suppose that f has a local minimum value at $x = c$.

$f(x) \ge f(c)$ for all values of x near enough to c.

$f(x) - f(c) \ge 0.$

Since c is an interior point of function domain,

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}.$$

$x \to c^+, x - c > 0$ *and* $\quad f(x) - f(c) \ge 0.$

$$f'(c) = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \ge 0. \qquad (3)$$

$x \to c^-, x - c < 0$ *and* $\quad f(x) - f(c) \ge 0.$

$$f'(c) = \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \le 0. \qquad (4)$$

By the equation (3) and (4), $f'(c) = 0$.

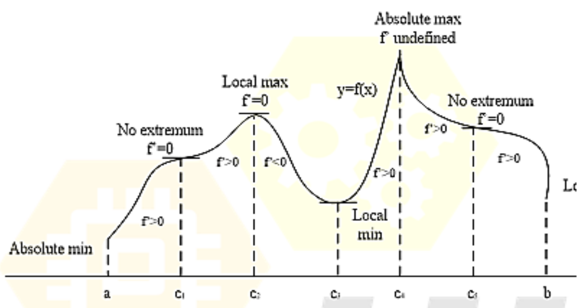

Figure 4. The critical points of a function locate where it is increasing and decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

V. SECOND DERIVATIVE TEST FOR LOCAL EXTREME

Suppose f'' is continuous on an open interval that contains $x = c$.

(1) If $f'(c) = 0$ *and* $f''(c) < 0$, then f has a local maximum at $x = c$.

(2) If $f'(c) = 0$ *and* $f''(c) > 0$, then f has a local manimum at $x = c$.

(3) If $f'(c) = 0$ *and* $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

VI. SOME EXAMPLE OF EXTREME VALUES

A. Example

The absolute maximum and minimum values of $f(\theta) = \sin\theta$ on the interval$[-\frac{\pi}{2}, \frac{5\pi}{6}]$ can be solved.

Differentiate with respect to in both sides,

$$f'(\theta) = \cos\theta.$$

To find the critical point , $f'(\theta) = 0$.

$$\cos\theta = 0,$$

$$\theta = \frac{\pi}{2}.$$

The critical value, $f(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$.

The endpoint value,$f(-\frac{\pi}{2}) = \sin(-\frac{\pi}{2}) = -1$

$$f(\frac{5\pi}{6}) = \sin(\frac{5\pi}{6}) = \frac{1}{2}.$$

$f(\theta)$ has the absolute maximum value is 1 at $\theta = \frac{\pi}{2}$ and the absolute minimum value is -1 at $\theta = -\frac{\pi}{2}$.

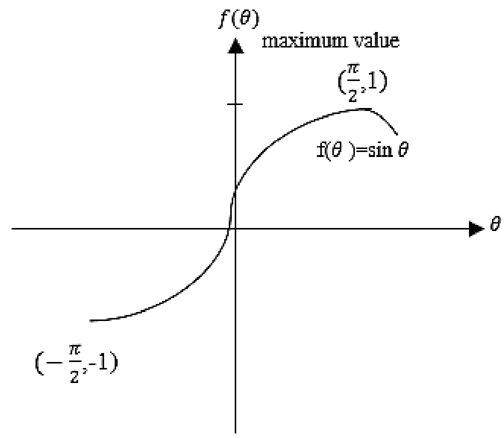

Figure 5. The absolute maximum and minimum value of sine function.

B. Example

The function $f(x) = (x^2 - 3)e^x$ has local maximum and local minimum values can be stated.

$$f(x) = (x^2 - 3)e^x.$$

Differentiate with respect to x in both sides,

$$f'(x) = (x^2 - 3)e^x + e^x(2x)$$

$$f'(x) = x^2e^x - 3e^x + 2x\,e^x$$

$$f'(x) = e^x(x^2 + 2x - 3).$$

Since e^x is never zero, the first derivative is zero, if and only if $x^2 + 2x - 3 = 0$.

$(x+3)(x-1) = 0$,

$x = -3, x = 1$.

The zeros x= - 3 and x=1 partition the x axis into intervals.

Interval	x<-3	-3<x<1	x>1
Sign of f'	+	−	+
Behavior of f	increasing	decreasing	increasing

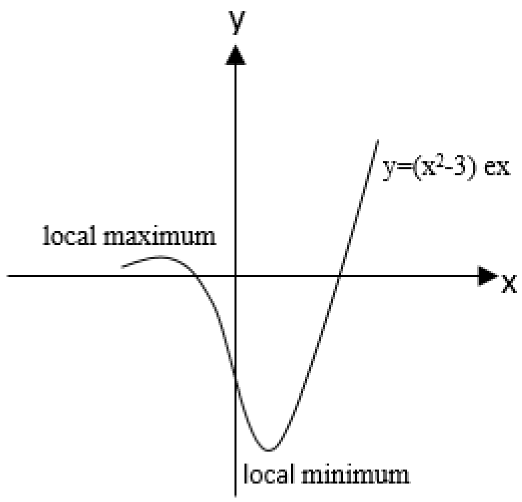

Figure 6.Local maximum and local minimum

Local maximum value is 0.299 at x= -3

Local minimum value is -5.437 at x=1.

VII. SOLVING APPLIED OPTIMIZATION PROBLEMS

A. Example

In a company, as a product which open-top box is to be made cutting small congruent square from the corners of the perimeter 40 inches that is each side has 10 inches' sheet of steel. If an open-top box is formed, extreme volume can be express and how large should the square cut from the corner of sheet steel.

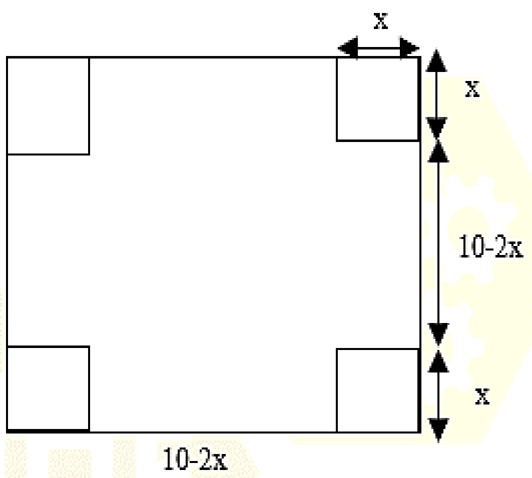

Figure 7. Steel Sheet

Let x be the side of the square cutting from each corners, l be the length , h be the high and w be the wide of the open-top . The volume of the box =V.

$V = lwh = (10-2x)(10-2x)x, 0 < x < 5.$

$V(x) = 100x - 40x^2 + 4x^3.$

$V'(x) = 100 - 80x + 12x^2.$

$V'(x) = 0 \rightarrow 100 - 80x + 12x^2 = 0.$

$4(25 - 20x + 3x^2) = 0$

$25 - 20x + 3x^2 = 0$

$(3x-5)(x-5) = 0,$

$x = \frac{5}{3}, x = 5.$

Only $x = \frac{5}{3}$ lies in the interior of the function domain .

Therefore, the critical point is $x = \frac{5}{3}$.

Critical value, $V\left(\frac{5}{3}\right) = 100\left(\frac{5}{3}\right) - 40\left(\frac{5}{3}\right)^2 + 4\left(\frac{5}{3}\right)^3$

$= 74.074.$

Endpoint value, $V(0) = 100(0) - 40(0) + 4(0)$

$V(0) = 0.$

$V(5) = 100(5) - 40(5)^2 + 4(5)^3$

$V(5) = 0.$

The maximum volume is 74.074 in^3.

The square cut from the corner of the sheet should be $\frac{5}{3}$ inches on the sides.

B. Example

We want to design, an open-top rectangular tank for catching rainwater. The tank is to be build 13500 ft^3 square based rectangular steel. The dimension for the base and high that will make the tank weight as little as possible can be found.

Let S be the surface area and

t be the thickness of the steel wall.

x be the length of the base side of tank.

y be the high of the tank.

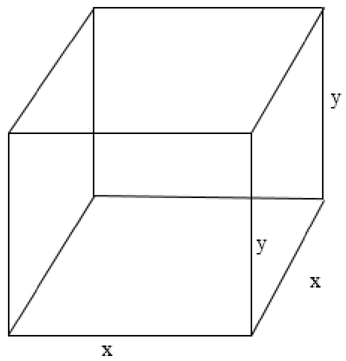

Figure 8. Open-top box

Surface area =area of base + area of four sides

$$S = x^2 + 4xy.$$

Weight of tank, W=thickness of steel x surface area

$W = t\ x\ S$

$Volume, V = x^2 y = 13500$

$$y = \frac{13500}{x^2}$$

$W = t(x^2 + 4xy)$

$W = t(x^2 + 4x\frac{13500}{x^2})$

$W = t(x^2 + \frac{54000}{x})$

$W' = t\left(2x - \frac{54000}{x^2}\right).$

$W'' = t(2 + \frac{108000}{x^3}).$

W' (x)=0→ $t\ \left(2x - \frac{54000}{x2}\right) = 0$

$$\left(2x - \frac{54000}{x2}\right) = 0$$

$$2x^3 - 54000 = 0$$

$$2x^3 = 54000$$

$$x^3 = 27000.$$

The critical pointis , x = 30 ft.

$$y = \frac{13500}{x^2} = \frac{13500}{900} = 15 \text{ ft.}$$

x=30→ $W'' = t\ (2 + \frac{108000}{27000}) = t(2 + 4) = 6\ t > 0.$

The process has minimum weight.

The dimension of base, x = 30 ft.

The dimension of high , y = 15 ft.

C. Example

In a steel company, as a product is designing a 200 liters the circular cylindrical shape can with steel wall. The dimension will use the least material can be computed.

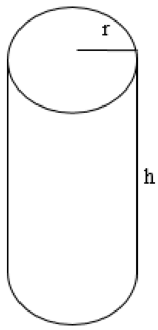

Figure 9. Circular Cylinder shape

Let V= volume of cylinder shape can.

V=200 liters =200,000 cm^3

Volume of cylinder shape can,

$V = \pi\, r^2 h$where V=volume, r=radius , h=high.

$\pi\, r^2 h$ =200,000

$$h = \frac{200{,}000}{\pi\, r^2}$$

Surface area of cylinder shape can , $A = 2\pi\, r^2 + 2\pi rh.$

Circle area is $\pi\, r^2$ and cylinder area is $2\pi rh$.

$$A = 2\pi\, r^2 + 2\pi r \frac{200{,}000}{\pi\, r^2}$$

$$A(r) = 2\pi\, r^2 + \frac{400{,}000}{r^2}$$

$$A'(r) = 4\pi r - \frac{400{,}000}{r^2}$$

$$A''(r) = 4\pi + \frac{800{,}000}{r^3}$$

$$A'(x) = 0{,}4\pi r - \frac{400000}{r^2} = 0$$

$$4\pi r^3 - 400000 = 0$$

$4\pi r^3$=400000

$$r^3 = \frac{400000}{4\pi}$$

$$r = \sqrt[3]{\frac{100000}{\pi}}$$

$$r = 31.69\ cm.$$

A" (31.69)=4 $\pi + \frac{800000}{(31.69)^3} = 37.7 > 0$ (minimum)

A (31.69)= $2\pi(31.69)^2 + \frac{400000}{31.69}$ = 18932.21 cm^2

The dimension of least material is

$$r = 31.69\ cm.$$

$$h = \frac{200000}{\pi r^2} = \frac{200000}{\pi(31.69)^2} = 63.4\ cm.$$

D. Example

A farmland rectangular shape will be bounded on one side by a stream and on the other three sides by a tin string fence with 120 ft. The largest area and its dimension can be solved.

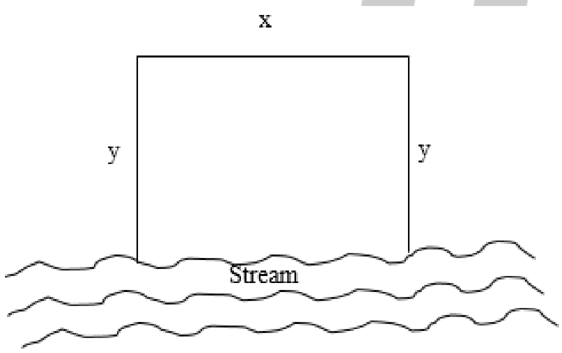

Figure 10. A farmland near the stream

Let x be length of the rectangular shape farmland.

y be the breath of rectangular shape farmland.

Perimeter of farmland=$x + 2y\ = 120ft$

$$y = \frac{120 - x}{2}$$

Area of rectangle shape farmland =A(x)

$$A(x) = xy$$

$$A(x) = x\text{- x } \frac{120 - x}{2}$$

$$= 60x - \frac{x^2}{2}$$

$$A'(x) = 60 - x$$

$$A''(x) = -1 < 0.$$

To find the critical point, A'(x) = 0.

60 – x = 0

x = 60 ft.

A"(x) = -1 < 0 (maximum).

The largest area, $A(x) = 3600 - \frac{3600}{2}$

$A(x)$ = 3600-1800

=1800 ft^2.

The length of farmland , x = 60 ft.

The breath of farmland, $y = \frac{120-x}{2} = \frac{120-60}{2}$ = 30 ft.

E. Example

A thanakha patch has 2400 ft^2 rectangular shape. It is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. The dimension for the outer rectangle will require the largest total length of fence can be computed. The amount of fence need can be found.

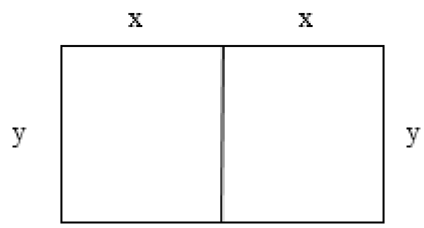

Figure 11. Thanakha patch rectangle shape

Let 2x be the length of the rectangle thanakha patch.

- y be the high of the rectangle thanakha patch.
- A be the area of the rectangle thanakha patch.
- P be the amount of fence need.

A=2400 ft^2

A=2xy=2400

y= $\frac{2400}{2x}$

y= $\frac{1200}{x}$.

$P = 4x + 3y.$

$P(x) = 4x + 3\frac{1200}{x}.$

$P(x) = 4x + \frac{3600}{x}.$

$P'(x) = 4 - \frac{3600}{x^2}.$

$P''(x) = \frac{7200}{x^3}$

$P'(x)=0 \rightarrow 4 - \frac{3600}{x^2}=0$

$4x^2$-3600=0

$x^2 = 900$

$x = \pm 30.$

Only x=30 lies in the interior of the function domain.

x=30→ $P''(30) = \frac{7200}{(30)^3} = \frac{7200}{27000} < 0.$

There is a maximum at x=30.

The dimension of the rectangle thanakha patch are

2x=60 ft and y = $\frac{1200}{30}$ =40 ft.

P=4x+3y =(120+120) = 240 ft

The amount of fence will be 240 ft needed.

VIII. CONCLUSION

We have use derivatives to find the extreme value of a process in social life. In the study of the differential equations, we should know critical points because it is main point to find the extreme values. We have expressed which how to compute the maximum values and the minimum values in social life. Thus, Engineering Mathematics is very importance language in our daily life. All of the people compute profit and loss of their business, mathematics, physics and economics by using the derivative and mathematical rules. We compute increase or decrease function during a interval in our social environment. We can find where the function's graph rises and falls and where any local extrema are located.

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